

# Markov type and threshold embeddings

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## Abstract

For two metric spaces  $X$  and  $Y$ , say that  $X$  *threshold-embeds into*  $Y$  if there exist a number  $K > 0$  and a family of Lipschitz maps  $\{\varphi_\tau : X \rightarrow Y : \tau > 0\}$  such that for every  $x, y \in X$ ,

$$d_X(x, y) \geq \tau \implies d_Y(\varphi_\tau(x), \varphi_\tau(y)) \geq \|\varphi_\tau\|_{\text{Lip}} \tau / K,$$

where  $\|\varphi_\tau\|_{\text{Lip}}$  denotes the Lipschitz constant of  $\varphi_\tau$ . We show that if a metric space  $X$  threshold-embeds into a Hilbert space, then  $X$  has Markov type 2. As a consequence, planar graph metrics and doubling metrics have Markov type 2, answering questions of Naor, Peres, Schramm, and Sheffield. More generally, if a metric space  $X$  threshold-embeds into a  $p$ -uniformly smooth Banach space, then  $X$  has Markov type  $p$ . Our results suggest some non-linear analogs of Kwapien's theorem. For instance, a subset  $X \subseteq L_1$  threshold-embeds into Hilbert space if and only if  $X$  has Markov type 2.

## 1 Introduction

We begin by recalling K. Ball's notion of Markov type [Bal92].

**Definition 1.1.** *A metric space  $(X, d)$  is said to have Markov type  $p \in [1, \infty)$  if there is a constant  $M > 0$  such that for every  $n \in \mathbb{N}$ , the following holds. For every reversible Markov chain  $\{Z_t\}_{t=0}^\infty$  on  $\{1, \dots, n\}$ , every mapping  $f : \{1, \dots, n\} \rightarrow X$ , and every time  $t \in \mathbb{N}$ ,*

$$\mathbb{E} d(f(Z_t), f(Z_0))^p \leq M^p t \mathbb{E} d(f(Z_0), f(Z_1))^p,$$

where  $Z_0$  is distributed according to the stationary measure of the chain. One denotes by  $M_p(X)$  the infimal constant  $M$  such that the inequality holds.

This is intended as a metrical generalization of the concept of *linear (Rademacher) type*, which we discuss shortly. One of Ball's primary motivations was in developing non-linear analog of Maurey's extension theorem for linear operators [Mau74]. Toward this end, he proved the following.

**Theorem 1.2** ([Bal92]). *Let  $(X, d)$  be a metric space and  $Y$  a Banach space. If  $X$  has Markov type 2 and  $Y$  is 2-uniformly convex, then there exists a constant  $C = C(X, Y)$  such that for every subset  $S \subseteq X$  and Lipschitz mapping  $f : S \rightarrow Y$ , there exists an extension  $\tilde{f} : X \rightarrow Y$  satisfying  $\tilde{f}|_S = f$  and  $\|\tilde{f}\|_{\text{Lip}} \leq C \|f\|_{\text{Lip}}$ .*

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Here, we use  $\|f\|_{\text{Lip}}$  to denote the infimal constant  $L$  such that  $f$  is  $L$ -Lipschitz. The preceding theorem is already very interesting in the case where  $Y$  is a Hilbert space. The notion of Markov type has since found a number of additional applications [LMN02, BLMN05, MN06, MN12].

Despite its apparent utility, only Hilbert spaces (and spaces which admit a bi-Lipschitz embedding into Hilbert space) were known to have Markov type 2 until the work of [NPSS06]. The authors prove that every  $p$ -uniformly smooth Banach has Markov type  $p$ . Most significantly, this implies that  $L_p$  for  $p > 2$  has Markov type 2, answering a fundamental open question. Combined with Ball's work, this gives a non-linear analog of Maurey's extension theorem in terms of uniform smoothness and convexity of the underlying Banach spaces. We refer to their work [NPSS06] for an extended discussion.

They also establish that trees and certain classes of Gromov hyperbolic spaces have Markov type 2. The authors state their belief that planar graph metrics and doubling metrics should have Markov type 2, but they are only able to show that such spaces have Markov type  $2 - \varepsilon$  for every  $\varepsilon > 0$ . Building on the method of [NPSS06], it can be shown that all series-parallel graph metrics have Markov type 2 [BKL07]. Both the planar and doubling questions have recently been reiterated in the survey of Naor [Nao12], where the author remarks that even the special case of the three-dimensional Heisenberg group  $\mathbb{H}^3$  is open. We resolve these questions and present a number of generalizations. Our main tool in controlling Markov type is the use of embeddings that are weaker than bi-Lipschitz.

**Threshold embeddings.** We recall that for two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , a mapping  $f : X \rightarrow Y$  is called *bi-Lipschitz* if  $f$  is invertible and both  $\|f\|_{\text{Lip}}$  and  $\|f^{-1}\|_{\text{Lip}}$  are bounded. The *distortion* of  $f$  is the quantity  $\|f\|_{\text{Lip}} \cdot \|f^{-1}\|_{\text{Lip}}$ . It is straightforward that Markov type is a bi-Lipschitz invariant. In fact, if there exists a map from  $X$  into  $Y$  with distortion  $D$ , then manifestly,  $M_p(X) \leq D \cdot M_p(Y)$ .

Unfortunately, there are planar graph metrics [Bou86, NR03, Laa02] and doubling metrics [Pan89, Sem96] (see also [LN06]) which do not admit any bi-Lipschitz embedding into a Hilbert space. However, such spaces are known to admit a weaker sort of embedding which we now recall. Say that  $X$  *threshold-embeds* into  $Y$  if there exists a constant  $K > 0$  and a family of mappings  $\{\varphi_\tau : X \rightarrow Y\}_{\tau > 0}$  such that the following holds: For every  $\tau > 0$ , for every  $x, y \in X$ ,

$$d_X(x, y) \geq \tau \implies d_Y(\varphi_\tau(x), \varphi_\tau(y)) \geq \frac{\|\varphi_\tau\|_{\text{Lip}}}{K} \tau.$$

If we wish to emphasize the constant  $K$ , we will say that  $X$   *$K$ -threshold-embeds* into  $Y$ . As opposed to bi-Lipschitz maps, threshold embeddings are only required to control one scale at a time. We prove the following theorem.

**Theorem 1.3.** *If  $X$  threshold-embeds into a Hilbert space, then  $X$  has Markov type 2. Quantitatively, if  $X$  admits a  $K$ -threshold-embedding, then  $M_2(X) \leq O(K)$ .*

Using the known constructions of threshold embeddings for various spaces (see Section 3.3), we confirm that a number of spaces have Markov type 2.

**Theorem 1.4.** *If  $(X, d)$  is the shortest-path metric on a weighted planar graph, then  $(X, d)$  has Markov type 2. More generally, this holds for the shortest-path metric on any surface of bounded genus. Quantitatively, if  $X$  is the shortest-path metric on a graph of orientable genus  $g > 1$ , then  $M_2(X) \leq O(\log g)$ .*

In Section 3.3, we show that this theorem generalizes even further, to any non-trivial minor-closed family of graphs.

We recall that a metric space  $(X, d)$  is said to be *doubling with constant  $\lambda$*  if every bounded set in  $X$  can be covered by  $\lambda$  sets of half the diameter. A space that is doubling with some constant  $\lambda < \infty$  is said to be *doubling*.

**Theorem 1.5.** *Every doubling metric space has Markov type 2. Quantitatively, if  $X$  is  $\lambda$ -doubling, then  $M_2(X) \leq O(\log \lambda)$ .*

Theorem 1.5 can be generalized; to this end, we now define the *Assouad-Nagata dimension* of a metric space  $(X, d)$ . This quantity, denoted  $\dim_{AN}(X, d)$ , is the least integer  $n$  such that the following holds: There exists a constant  $c > 0$  so that for every number  $r > 0$ , there is a cover  $X \subseteq \bigcup_{i=1}^{\infty} U_i$  of  $X$  such that each set  $U_i$  has  $\text{diam}(U_i) \leq cr$  and every ball of radius  $r$  in  $X$  has non-trivial intersection with at most  $n + 1$  elements of  $\{U_i\}_{i=1}^{\infty}$ . Spaces of bounded Assouad-Nagata dimension include trees, Gromov hyperbolic groups, manifolds of pinched negative sectional curvature, Euclidean buildings, and homogeneous Hadamard manifolds (see [LS05]). In Section 3.3, we prove the following.

**Theorem 1.6.** *If  $(X, d)$  is a metric space with finite Assouad-Nagata dimension, then  $X$  has Markov type 2.*

**Uniformly smooth Banach spaces.** The modulus of uniform smoothness of a Banach space  $X$  is defined, for  $\varepsilon > 0$ , as

$$\rho_X(\varepsilon) = \sup \left\{ \frac{\|x + \varepsilon y\| + \|x - \varepsilon y\|}{2} - 1 : x, y \in X, \|x\| = \|y\| = 1 \right\}.$$

The space  $X$  is called *uniformly smooth* if  $\lim_{\varepsilon \rightarrow 0} \frac{\rho_X(\varepsilon)}{\varepsilon} = 0$ . Furthermore,  $X$  is said to be  *$p$ -uniformly smooth* if there is a constant  $S > 0$  such that  $\rho_X(\varepsilon) \leq S^p \varepsilon^p$  for all  $\varepsilon > 0$ . We use  $S_p(X)$  to denote the infimal constant  $S$  for which this holds. It can be verified that a Banach space can only be  $p$ -uniformly smooth for  $p \leq 2$ . Furthermore,  $L_p$  is  $p$ -uniformly smooth for  $1 \leq p \leq 2$  and 2-uniformly smooth for  $p \geq 2$  [Han56]; see also [BL00, App. A]. We extend Theorem 1.3 to spaces which threshold-embed into  $p$ -uniformly smooth Banach spaces.

**Theorem 1.7.** *If a metric space  $(X, d)$  threshold-embeds into a  $p$ -uniformly smooth Banach space then  $X$  has Markov type  $p$ . Quantitatively if  $X$  admits a  $K$ -threshold-embedding into a  $p$ -uniformly smooth space  $Y$ , then  $M_p(X) \leq O(K S_p(Y))$ .*

**Linear type, cotype, and Kwapien's theorem.** We now review some fundamental definitions from the geometry of Banach spaces. A Banach space  $X$  is said to have (Rademacher) type  $p > 0$  if there exists a constant  $T > 0$  such that for all finite sequences  $x_1, \dots, x_n \in X$ ,

$$\mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^p \leq T^p \sum_{i=1}^n \|x_i\|^p,$$

where  $\{\varepsilon_1, \dots, \varepsilon_n\}$  is an i.i.d. sequence of random signs. The least such constant  $T$  is referred to as the type  $p$  constant of  $X$  and denoted  $T_p(X)$ . Similarly,  $X$  is said to have (Rademacher) cotype

$q < \infty$  if there exists a constant  $C > 0$  such that for all finite sequences  $x_1, \dots, x_n \in X$ ,

$$\mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^q \geq \frac{1}{C^q} \sum_{i=1}^n \|x_i\|^q.$$

The least such constant  $C$  is denoted  $C_q(X)$  and called the cotype  $q$  constant of  $X$ .

It is straightforward that any Hilbert space  $H$  has type 2 and cotype 2 and, in fact,  $T_2(H) = C_2(H) = 1$ . A fundamental theorem of Kwapien states that, up to isomorphism, Hilbert space is the only Banach space with these properties.

**Theorem 1.8** ([Kwa72]). *A Banach space has type 2 and cotype 2 if and only if it is linearly isomorphic to a Hilbert space. Furthermore, the isomorphism constant is bounded by  $T_2(X) \cdot C_2(X)$ .*

In line with the “Ribe program” (see, e.g., [Nao12]) and the local theory of Banach spaces, one might look for non-linear analogs of Kwapien’s result. This would involve non-linear notions of type (e.g., [Bal92]) and cotype (e.g., [MN08]) and a replacement of “linear isomorphism” by a suitable non-linear generalization. We refer to [BL00, MN08, Nao12] for a thorough discussion of related issues. Unfortunately, it is folklore that the most natural generalization, in which one replaces a linear isomorphism by a bi-Lipschitz mapping is patently false.

To see this, note that there is a family of graph metrics  $\{G_k\}_{k=0}^\infty$  called the *Laakso graphs* (after [Laa02]) which bi-Lipschitz embed into  $L_1$  with distortion at most 2 [GNRS04]. Furthermore, it is proved that these graphs have Markov type 2 with uniform constant [NPSS06], i.e.  $\sup_{k \geq 1} M_2(G_k) < \infty$ . Since  $L_1$  has cotype 2 (see, e.g., [LT79]), a straightforward non-linear Kwapien theorem would state that they admit bi-Lipschitz embeddings into a Hilbert space with uniformly bounded distortion, but this is known to be impossible [Laa02]. Note that one can easily construct a single infinite subset of  $L_1$  (by taking a suitable infinite union of the graphs) which has Markov type 2 but admits no bi-Lipschitz embedding into a Hilbert space.

We propose that the correct analog of “linear isomorphism” in the setting of Kwapien’s theorem is the notion of a threshold embedding. To this end, one should first observe the following.

**Theorem 1.9.** *A Banach space  $X$  threshold-embeds into a Hilbert space if and only if it is linearly isomorphic to a Hilbert space.*

Initially, we proved this using Theorem 1.3, but Assaf Naor pointed out to us that it follows in a simpler way using the notion of Enflo type, which was introduced in [Enf70]. A metric space  $(X, d)$  is said to have *Enflo type  $p$*  if there is a constant  $E > 0$  such that for every  $n \in \mathbb{N}$  and every mapping  $f : \{0, 1\}^n \rightarrow X$ , we have the inequality

$$\sum_{\substack{x, y \in \{0, 1\}^n \\ \|x - y\|_1 = n}} d(f(x), f(y))^p \leq E^p \sum_{\substack{x, y \in \{0, 1\}^n \\ \|x - y\|_1 = 1}} d(f(x), f(y))^p.$$

In this case, one says that  $(X, d)$  has Enflo type  $p$  with constant  $E$ . It is known that for any metric space, Markov type  $p$  implies Enflo type  $p$  [NS02, Prop. 1].

**Proposition 1.10** (Naor, personal communication). *Suppose  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces and  $Y$  has Enflo type  $p$  with constant  $E$ . If  $X$   $K$ -threshold-embeds into  $Y$ , then  $X$  has Enflo type  $p$  with constant  $O(KE)$ .*

Using the preceding result along with a theorem of [AMM85] and Kwapien's theorem, Theorem 1.9 follows readily. We refer to Section 5.

While we are not able to give a full non-linear analog of Kwapien's theorem, we do take some steps in this direction. In particular, our methods are strong enough to give such a result for subsets of  $L_1$ . Recall that a *uniform embedding*  $f : X \rightarrow Y$  between metric spaces  $X$  and  $Y$  is an invertible map such that  $f$  and  $f^{-1}$  are both uniformly continuous. It is known that if a Banach space  $X$  admits a uniform embedding into a Hilbert space, then  $X$  has cotype 2, but the converse is not true, even for spaces of non-trivial type (see [BL00, §8.2]). We prove the following first step.

**Theorem 1.11.** *Suppose that  $X$  is a Banach space that admits a uniform embedding into a Hilbert space. Then a subset  $S \subseteq X$  threshold-embeds into Hilbert space if and only if  $S$  has Markov type 2.*

In particular, it is well-known that  $L_1$  uniformly embeds into  $L_2$ , thus a subset of  $L_1$  threshold-embeds into Hilbert space if and only if it has Markov type 2. There seem to be two non-trivial steps in completing a non-linear Kwapien theorem. The first involves metric subsets of linear spaces.

**Conjecture 1.12.** *If  $X$  is a Banach space of cotype 2, then a subset  $S \subseteq X$  threshold-embeds into Hilbert space if and only if  $S$  has Markov type 2.*

We are not able to resolve the validity of the conjecture for the spaces of Schatten class operators  $C_p$  for  $1 \leq p < 2$  (see [BL00, §8.2]). A truly satisfactory non-linear Kwapien theorem would involve no reference to linear spaces at all; it would instead rely on an appropriate notion of metric cotype.

**Question 1.13.** *Suppose that  $(X, d)$  is a metric space of metric cotype 2 (in the sense of [MN06]) and Markov type 2. Does this imply that  $X$  threshold-embeds into Hilbert space?*

Part of the other side of this question is open as well. By the non-linear Maurey-Pisier Theorem [MN06], we know that if  $X$  threshold-embeds into a Hilbert space then it has finite metric cotype, but more should be true.

**Question 1.14.** *If  $(X, d)$  threshold-embeds into Hilbert space, does this imply that  $X$  has metric cotype 2?*

This would imply, in particular, that planar and doubling metrics have metric cotype 2, answering a question of [Nao12, §4].

**Finite metric spaces and some historical remarks.** Threshold embeddings have been studied in the context of embeddings of finite metric spaces into Banach spaces. Rao [Rao99] showed that finite planar graph metrics admit  $O(1)$ -threshold-embeddings into Euclidean space in his proof of a multi-commodity max-flow/min-cut theorem. More generally, using the results of [KPR93], he proved this for any family of graphs excluding a fixed minor. On the other hand, Bourgain [Bou86] exhibited an infinite family of (finite) trees that admit no bi-Lipschitz embedding with uniformly bounded distortion into a Hilbert space.

It was earlier shown by Semmes [Sem96], using an important result of Pansu [Pan89], that the 3-dimensional Heisenberg group  $\mathbb{H}^3$  (which is doubling) does not admit a bi-Lipschitz embedding into any finite-dimensional Euclidean space. Since Pansu's technique can be extended to any Banach space having the Radon-Nikodym property (see [LN06]), this yields a metric space which threshold-embeds into Hilbert space, but does not bi-Lipschitz embed into any Banach space with

the RNP. An example of [Laa02] gives an infinite doubling, planar graph metric which does not bi-Lipschitz embed into any uniformly convex Banach space. It was an open problem (of relevance to applications in theoretical computer science) to determine whether any metric space which threshold-embeds into Hilbert space admits a bi-Lipschitz embedding into  $L_1$ . This was resolved negatively by Cheeger and Kleiner [CK10] who showed that  $\mathbb{H}^3$  does not bi-Lipschitz embed into  $L_1$ .

Returning to finite metric spaces, the utility of a family of mappings for each scale was made explicit in [KLMN05] where the authors use this approach to give a new proof of Bourgain’s theorem [Bou85] on embedding of finite metric spaces into Hilbert space. In [Lee05], it is proved that if an  $n$ -point metric space  $X$   $K$ -threshold-embeds into  $L_p$  for  $p \geq 2$ , then  $X$  bi-Lipschitz embeds into  $L_p$  with distortion  $O(K^{1-1/p}(\log n)^{1/p})$ . In particular, such a space admits a distortion  $O(K)$  embedding into  $L_p$  for some  $p = O(\log \log n)$ .

In [NPSS06], what we call threshold embeddings are referred to as “weak embeddings.” The authors also define a notion of “weak Markov type 2” and show that this property readily follows from the existence of a threshold embedding into Hilbert space.

## 1.1 Outline of our approach

Let  $\mathcal{Z}$  be a normed space and consider a Markov chain  $\{Z_t\}_{t=0}^\infty$  on a finite state space  $\Omega$  and a mapping  $f : \Omega \rightarrow \mathcal{Z}$ . In [NPSS06], it is shown that for every time  $t \in \mathbb{N}$ , there exist martingales  $\{M_k\}_{k=0}^t$  and  $\{N_k\}_{k=0}^t$  such that

$$f(Z_{2t}) - f(Z_0) = M_t - N_t, \quad (1)$$

where  $\{M_k\}$  and  $\{N_k\}$  both naturally trace the evolution of  $\{f(Z_k)\}$  forward in time and backward in time, respectively. The decomposition is reviewed in Section 3.1. This allows one to reduce various problems on Markov chains to potentially easier problems on martingales.

Now consider a metric space  $(X, d)$  and a threshold embedding  $\{\varphi_\tau : X \rightarrow \mathcal{Z} : \tau > 0\}$  of  $X$  into  $\mathcal{Z}$ . In determining the Markov type of  $(X, d)$ , one is naturally led to study mappings  $g : \Omega \rightarrow X$  via the composition maps  $\varphi_\tau \circ g : \Omega \rightarrow \mathcal{Z}$ . But crucially, the martingales  $\{M_k\}$  and  $\{N_k\}$  from (1) depend heavily on the map  $f$ , and thus the problem of deducing Markov type from a threshold embedding becomes one of controlling an entire family of martingales—one for every map  $\varphi_\tau \circ g$  for  $\tau > 0$ .

Fortunately, when one allows the map  $f$  in (1) to vary, all the martingales that arise are defined with respect to the same pair of filtrations, and their differences can be uniformly controlled in terms of jumps of the chain  $\{f(Z_k)\}$ . This leads to a problem on simultaneously bounding the tail of all martingales whose differences are subordinate to a common sequence of random variables. We present a representative lemma of this form, for the case of real-valued martingales.

**Lemma 1.15.** *There exists a constant  $K > 0$  such that the following holds. Let  $\{\mathcal{F}_t\}$  be a filtration, and let  $\{\alpha_t\}$  be a sequence adapted to  $\{\mathcal{F}_t\}$ . Let*

$$\left\{ \{M_t^\xi\} : \xi \in I \right\}$$

*be a countable collection of real-valued martingales with respect to  $\{\mathcal{F}_t\}$  such that  $|M_t^\xi - M_{t-1}^\xi| \leq \alpha_t$  for all  $t \geq 1$ . Then for every  $n \geq 0$ , we have*

$$\int_0^\infty y \sup_{\xi \in I} \mathbb{P}(|M_n^\xi - M_0^\xi| \geq y) dy \leq K \sum_{t=1}^n \mathbb{E}(\alpha_t^2).$$

A generalization of this result to martingales taking values in uniformly smooth Banach spaces appears as Lemma 2.6. Our approach to Lemma 1.15 is via some classical distributional inequalities of Burkholder and Gundy [BG70] (see Theorem 2.1) which allow one to control the maximum process associated to certain martingales in terms of the corresponding square functions.

We extend Lemma 1.15 to Hilbert-space-valued martingales using the martingale dimension reduction of Kallenberg and Sztencel [KS91] (see Lemma 2.3). A version for  $p$ -uniformly smooth Banach spaces requires a non-trivial extension of their technique appearing in Section 4. The intuition that an analogous bound should hold for  $p$ -uniformly smooth spaces goes back to work of Pisier [Pis75], who characterized uniform smoothness in terms of certain martingale inequalities. We state our “dimension reduction” lemma here as it may be of independent interest. We use ‘ $\preceq$ ’ to denote stochastic domination.

**Lemma 1.16.** *For  $p \in (1, 2]$ , the following holds. Let  $\mathcal{Z}$  be a  $p$ -uniformly smooth Banach space and let  $\{M_t\}$  be a  $\mathcal{Z}$ -valued martingale with respect to the filtration  $\{\mathcal{F}_t\}$ . Then there exists an  $\mathbb{R}^2$ -valued martingale  $\{N_t\}$  and a constant  $K > 0$  such that for any time  $t \geq 0$ , the following holds.*

$$i) \|M_t - M_0\|^p \preceq \|N_t - N_0\|_2^2, \text{ and}$$

$$ii) \|N_{t+1} - N_t\|_2^2 \preceq K (\|M_{t+1} - M_t\|^p + \mathbb{E}[\|M_{t+1} - M_t\|^p \mid \mathcal{F}_{t-1}]),$$

where  $K = O(S_p(\mathcal{Z})^p)$ .

With these tools in hand, the proof of Theorem 1.7 is carried out in Section 3.2. In Section 3.3, we recall how one constructs threshold-embeddings into Hilbert space using random partitions, yielding the proofs of Theorems 1.4, 1.5, and 1.6. Finally, in Section 5 we discuss some issues around a non-linear version of Kwapien’s theorem.

## 2 Distributional inequalities for martingales

We now study the uniform tail behavior of martingales whose difference sequences are subordinate to a common sequence of random variables. In the next section, we address the case of  $\mathbb{R}$ -valued martingales using a distributional inequality of [BG70]. In Section 2.2, we extend these bounds to martingales taking values in a uniformly smooth Banach space.

### 2.1 One-dimensional martingales

Consider a map  $\Phi : [0, \infty) \rightarrow [0, \infty)$ . Say that  $\Phi$  is  $\gamma$ -doubling for some  $\gamma > 0$  if it is continuous and non-decreasing, with  $\Phi(0) = 0$ , and satisfies, for every  $x > 0$ ,

$$\Phi(2x) \leq \gamma \Phi(x). \quad (2)$$

For a real-valued martingale  $\{M_t\}_{t=0}^n$ , we write  $M_t^* = \max_{0 \leq s \leq t} |M_s|$ . The next theorem appears as [BG70, Thm. 5.2] (see also [Bur73, Thm. 11.1]).

**Theorem 2.1.** *For every  $\delta > 0$  and  $\gamma > 0$ , there exist constants  $C, c > 0$  (depending on  $\delta$  and  $\gamma$ ) such that the following holds. Let  $\{M_t\}_{t=0}^n$  be a martingale relative to the filtration  $\{\mathcal{F}_t\}$  such that  $M_0 = 0$ . Suppose that, for every  $1 \leq t \leq n$ ,*

$$\mathbb{E}[|M_t - M_{t-1}| \mid \mathcal{F}_{t-1}] \geq \delta \cdot (\mathbb{E}[|M_t - M_{t-1}|^2 \mid \mathcal{F}_{t-1}])^{1/2}. \quad (3)$$

Then for any  $\Phi : [0, \infty) \rightarrow [0, \infty)$  which is  $\gamma$ -doubling, one has

$$c\mathbb{E}\Phi(S_n) \leq \mathbb{E}\Phi(M_n^*) \leq C\mathbb{E}\Phi(S_n),$$

where  $S_n^2 := \sum_{t=1}^n |M_t - M_{t-1}|^2$ .

We will use the preceding theorem to get uniform control on families of martingales in terms of their square functions.

**Proposition 2.2.** *Consider a martingale  $\{M_t\}$  with respect to the filtration  $\{\mathcal{F}_t\}$ . Let*

$$\Gamma_n^2 := \sum_{t=1}^n \left( |M_t - M_{t-1}| + \sqrt{\mathbb{E}[|M_t - M_{t-1}|^2 \mid \mathcal{F}_{t-1}]} \right)^2.$$

Then there exists an absolute constant  $C > 0$  such that for all  $\theta \geq 0$ ,

$$\mathbb{P}(|M_n - M_0| \geq \theta) \leq C\mathbb{E}\Phi_\theta(\Gamma_n),$$

where  $\Phi_\theta(x) := (x/\theta)^3 \wedge 1$ .

*Proof.* We may assume that  $M_0 = 0$ . In order to apply Theorem 2.1, we construct a modified martingale. For  $t \geq 0$ , let  $\sigma_t := (\mathbb{E}[|M_t - M_{t-1}| \mid \mathcal{F}_{t-1}])^{1/2}$ , and let  $\{\varepsilon_t\}_{t \geq 0}$  be an independent sequence of i.i.d.  $\pm 1$  Bernoulli random variables. Now define a new martingale,

$$\tilde{M}_t := M_t + \sum_{i=1}^t \varepsilon_i \sigma_i, \tag{4}$$

and observe that  $\tilde{S}_n^2 := \sum_{t=1}^n |\tilde{M}_t - \tilde{M}_{t-1}|^2 \leq \Gamma_n^2$ .

Clearly we have, for every  $1 \leq t \leq n$ ,

$$\mathbb{E} \left[ |\tilde{M}_t - \tilde{M}_{t-1}| \mid \mathcal{F}_{t-1} \right] \geq \sigma_t = \frac{1}{\sqrt{2}} \left( \mathbb{E} \left[ |\tilde{M}_t - \tilde{M}_{t-1}|^2 \mid \mathcal{F}_{t-1} \right] \right)^{1/2},$$

thus  $\{\tilde{M}_t\}_{t=0}^n$  satisfies (3) with  $\delta = 1/\sqrt{2}$ .

From the definition of  $\Phi_\theta$ , we have

$$\mathbb{P}(|\tilde{M}_n| \geq \theta) \leq \mathbb{E}\Phi_\theta(|\tilde{M}_n|). \tag{5}$$

Since  $\Phi_\theta$  is clearly 8-doubling, we can apply Theorem 2.1 and conclude that for some  $C > 0$ ,

$$\mathbb{E}\Phi_\theta(|\tilde{M}_n|) \leq C\mathbb{E}\Phi_\theta(\tilde{S}_n) \leq C\mathbb{E}\Phi_\theta(\Gamma_n). \tag{6}$$

We are thus left to relate  $M_n$  and  $\tilde{M}_n$ .

Note that from (4), we have

$$\begin{aligned} \{\tilde{M}_n \geq \theta\} &\supset \{M_n \geq \theta\} \cap \{\sum_{t=1}^n \varepsilon_t \sigma_t \geq 0\}, \\ \{\tilde{M}_n \leq -\theta\} &\supset \{M_n \leq -\theta\} \cap \{\sum_{t=1}^n \varepsilon_t \sigma_t \leq 0\}. \end{aligned}$$

But now independence and symmetry imply that

$$\mathbb{P}(|M_n| \geq \theta) \leq 2\mathbb{P}(|\tilde{M}_n| \geq \theta) \leq 2C\mathbb{E}\Phi_\theta(\Gamma_n),$$

where the final inequality follows from (5) and (6).  $\square$



## 2.2 Martingales in smooth Banach space

We first extend Proposition 2.2 to martingales taking values in a Hilbert space. We will need the following martingale dimension reduction lemma which is a special case [KS91] who prove it in the more difficult setting of continuous-time martingales. For an exposition of the discrete case, see [KW92, Prop. 5.8.3].

**Lemma 2.3.** *Let  $\{M_t\}$  be an  $L_2$ -valued martingale. Then there exists an  $\mathbb{R}^2$ -valued martingale  $\{N_t\}$  such that for any time  $t \geq 0$ ,  $\|N_t\|_2 = \|M_t\|_2$  and  $\|N_{t+1} - N_t\|_2 = \|M_{t+1} - M_t\|_2$ .*

This immediately yields the following analog to Proposition 2.2.

**Corollary 2.4.** *Consider an  $L_2$ -valued martingale  $\{M_t\}$  with respect to the filtration  $\{\mathcal{F}_t\}$ . Let*

$$\Gamma_n^2 := \sum_{t=1}^n \left( \|M_t - M_{t-1}\| + \sqrt{\mathbb{E}[\|M_t - M_{t-1}\|^2 \mid \mathcal{F}_{t-1}]} \right)^2.$$

*Then there exists an absolute constant  $C > 0$  such that for all  $\theta \geq 0$ ,*

$$\mathbb{P}(\|M_n - M_0\| \geq \theta) \leq C\mathbb{E}\Phi_\theta(\Gamma_n),$$

*where  $\Phi_\theta(x) := (x/\theta)^3 \wedge 1$ .*

*Proof.* Consider the corresponding  $\mathbb{R}^2$ -valued martingale  $\{N_t\}$  from Lemma 2.3. Applying Proposition 2.2 to each coordinate separately yields the desired claim.  $\square$

In Section 4, we will prove the following generalization to uniformly smooth spaces.

**Lemma 2.5.** *For  $p \in (1, 2]$ , let  $\mathcal{Z}$  be a  $p$ -uniformly smooth Banach space. Consider a  $\mathcal{Z}$ -valued martingale  $\{M_t\}$  with respect to the filtration  $\{\mathcal{F}_t\}$ . Let*

$$\Gamma_n^2 := \sum_{t=1}^n \left( \|M_t - M_{t-1}\|^{p/2} + \sqrt{\mathbb{E}[\|M_t - M_{t-1}\|^p \mid \mathcal{F}_{t-1}]} \right)^2.$$

*Then there exists an absolute constant  $C > 0$  and a number  $C' = O(S_p(\mathcal{Z})^{p/2})$  such that for all  $\theta \geq 0$ ,*

$$\mathbb{P}(\|M_n - M_0\|^{p/2} \geq \theta) \leq C\mathbb{E}\Phi_\theta(C'\Gamma_n),$$

*where  $\Phi_\theta(x) := (x/\theta)^3 \wedge 1$ .*

*Proof.* Construct the 2-dimensional martingale  $\{N_t\}$  as in Lemma 1.16. Now apply Proposition 2.2 to each of the coordinates of  $N_t$  to arrive at the desired conclusion.  $\square$

The preceding lemma leads to the following consequence on uniform integrability of martingales whose differences are subordinate to a common sequence of random variables. Readers interested primarily in the Hilbert space case should note that for  $p = 2$  and  $\mathcal{Z} = L_2$ , we only require Corollary 2.4.

**Lemma 2.6.** For  $p \in (1, 2]$ , let  $\mathcal{Z}$  be a  $p$ -uniformly smooth Banach space. There exists a constant  $K = O(S_p(\mathcal{Z})^p)$  such that the following holds. Let  $\{\mathcal{F}_t\}$  be a filtration, and let  $\{\alpha_t\}$  be a sequence adapted to  $\{\mathcal{F}_t\}$ . Let

$$\left\{ \{M_t^\xi\} : \xi \in I \right\}$$

be a countable collection of  $\mathcal{Z}$ -valued martingales with respect to  $\{\mathcal{F}_t\}$  such that  $\|M_t^\xi - M_{t-1}^\xi\| \leq \alpha_t$  for all  $t \geq 1$ . Then for every  $n \geq 0$ , we have

$$\int_0^\infty y^{p-1} \sup_{\xi \in I} \mathbb{P}(\|M_n^\xi - M_0^\xi\| \geq y) dy \leq K \sum_{t=1}^n \mathbb{E}(\alpha_t^p).$$

*Proof.* We may assume that  $M_0^\xi = 0$  for all  $\xi \in I$ . Let  $\Gamma_n^2 := \sum_{t=1}^n \left( |\alpha_t|^{p/2} + \sqrt{\mathbb{E}(\alpha_t^p | \mathcal{F}_{t-1})} \right)^2$  and for each  $\xi \in I$ , define

$$(\Gamma_n^\xi)^2 := \sum_{t=1}^n \left( \|M_t^\xi - M_{t-1}^\xi\|^{p/2} + \sqrt{\mathbb{E}[\|M_t^\xi - M_{t-1}^\xi\|^p | \mathcal{F}_{t-1}]} \right)^2.$$

Then from our assumptions, for any  $\xi \in I$ , we have  $\Gamma_n^\xi \leq \Gamma_n$ .

Put  $A_n^2 := \sum_{t=1}^n \alpha_t^p$  and for  $\lambda > 0$ , define  $\Phi_\lambda(x) := (x/\lambda\sqrt{\mathbb{E}A_n^2})^3 \wedge 1$ . Applying Lemma 2.5, we obtain that there exist an absolute constant  $C > 0$  and a number  $C' = O(S_p(\mathcal{Z})^{p/2})$  such that for any  $\xi \in I$ ,

$$\mathbb{P}\left(\|M_n^\xi\|^{p/2} \geq \lambda\sqrt{\mathbb{E}A_n^2}\right) \leq C\mathbb{E}\Phi_\lambda(C'\Gamma_n^\xi) \leq C\mathbb{E}\Phi_\lambda(C'\Gamma_n). \quad (7)$$

Let  $\mu$  be the distribution of  $C\Gamma_n$ . Then we have,

$$\begin{aligned} \int_0^\infty y^{p-1} \sup_{\xi \in I} \mathbb{P}(\|M_n^\xi\| \geq y) dy &= \frac{2}{p} \int_0^\infty y \sup_{\xi \in I} \mathbb{P}(\|M_n^\xi\|^{p/2} \geq y) dy \\ &= \frac{2}{p} \mathbb{E}(A_n^2) \int_0^\infty \lambda \sup_{\xi \in I} \mathbb{P}\left(\|M_n^\xi\|^{p/2} \geq \lambda\sqrt{\mathbb{E}A_n^2}\right) d\lambda \\ &\stackrel{(7)}{\leq} 2C\mathbb{E}(A_n^2) \int_0^\infty \lambda \mathbb{E}\Phi_\lambda(C'\Gamma_n) d\lambda \\ &= 2C\mathbb{E}(A_n^2) \int_0^\infty \lambda \int_0^\infty \left( \frac{x^3}{\lambda^3(\mathbb{E}A_n^2)^{3/2}} \wedge 1 \right) d\mu(x) d\lambda \\ &= 2C\mathbb{E}(A_n^2) \int_0^\infty \left( \int_0^{x\sqrt{\mathbb{E}A_n^2}} \lambda d\lambda + \int_{x\sqrt{\mathbb{E}A_n^2}}^\infty \left( \frac{x^3}{\lambda^2(\mathbb{E}A_n^2)^{3/2}} \right) d\lambda \right) d\mu(x) \\ &= 2C \int_0^\infty \frac{x^2}{2} + x^2 d\mu(x) \\ &= 3C(C'^2)\Gamma_n^2. \end{aligned}$$

Combined with the fact that  $\mathbb{E}(\Gamma_n^2) \leq 4\mathbb{E}(A_n^2)$ , this completes the proof.  $\square$

### 3 Markov type and threshold embeddings

With Lemma 2.6 in hand, we are ready to prove our main theorem. We first review the decomposition of a Markov chain on a normed space into a pair of martingales. Then in Section 3.2, we relate Markov type and threshold embeddings into uniformly smooth spaces. Finally, in Section 3.3, we use this to endow certain families of metric spaces with Markov type 2.

#### 3.1 The martingale decomposition

Let  $\mathcal{Z}$  be a normed space, and let  $\{Z_s\}_{s=0}^\infty$  be a stationary, reversible Markov chain on a finite state space  $\Omega$ . Consider any  $f : \Omega \rightarrow \mathcal{Z}$  and  $t \in \mathbb{N}$ . Define the martingales  $\{M_s\}_{s=0}^t$  and  $\{N_s\}_{s=0}^t$  by  $M_0 = f(Z_0)$  and  $N_0 = f(Z_t)$  and for  $0 \leq s \leq t-1$ ,

$$M_{s+1} - M_s := f(Z_{s+1}) - f(Z_s) - \mathbb{E}[f(Z_{s+1}) - f(Z_s) \mid Z_s] \quad (8)$$

$$N_{s+1} - N_s := f(Z_{t-s+1}) - f(Z_{t-s}) - \mathbb{E}[f(Z_{t-s+1}) - f(Z_{t-s}) \mid Z_{t-s}]. \quad (9)$$

Observe that  $\{M_s\}$  is a martingale with respect to the filtration induced on  $\{Z_0, Z_1, \dots, Z_t\}$  and  $\{N_s\}$  is a martingale with respect to the filtration induced on  $\{Z_t, Z_{t-1}, \dots, Z_0\}$ . For every  $1 \leq s \leq t-1$ , it is elementary to verify that

$$f(Z_{s+1}) - f(Z_{s-1}) = (M_{s+1} - M_{s-1}) - (N_{t-s+1} - N_{t-s}). \quad (10)$$

We also define the martingales  $\{A_k\}_{0 \leq k \leq t/2}$  and  $\{B_k\}_{0 \leq k \leq t/2}$  by

$$A_k := \sum_{s=1}^k M_{2s} - M_{2s-1}$$

$$B_k := \sum_{s=1}^k N_{2s} - N_{2s-1}.$$

If we assume that  $t$  is even, then summing (10) over  $s = 1, 3, 5, \dots, t/2 - 1$  yields

$$f(Z_t) - f(Z_0) = A_{t/2} - B_{t/2}. \quad (11)$$

Finally, we observe that for any  $1 \leq s \leq t/2$  and  $p \geq 1$ , we have the inequalities

$$\|A_s - A_{s-1}\|^p \leq 2^{p-1} \|f(Z_{2s}) - f(Z_{2s-1})\|^p + 2^{p-1} \mathbb{E} \left[ \|f(Z_{2s}) - f(Z_{2s-1})\|^p \mid Z_{2s-1} \right] \quad (12)$$

$$\|B_s - B_{s-1}\|^p \leq 2^{p-1} \|f(Z_{t-2s+1}) - f(Z_{t-2s})\|^p + 2^{p-1} \mathbb{E} \left[ \|f(Z_{t-2s+1}) - f(Z_{t-2s})\|^p \mid Z_{t-2s} \right] \quad (13)$$

These follow from  $A_s - A_{s-1} = M_{2s} - M_{2s-1}$  and  $B_s - B_{s-1} = N_{2s} - N_{2s-1}$  along with the definitions (8) and (9).

#### 3.2 Threshold embeddings

**Theorem 3.1.** *For  $p \in (1, 2]$ , let  $\mathcal{Z}$  be a  $p$ -uniformly smooth Banach space. If  $(X, d)$  is a metric space that threshold-embeds into  $\mathcal{Z}$ , then  $X$  has Markov type  $p$ . Quantitatively, if  $X$   $D$ -threshold-embeds into  $\mathcal{Z}$ , then  $M_p(X) \leq O(DS_p(\mathcal{Z}))$ .*

*Proof.* Let  $\{\varphi_\tau : X \rightarrow \mathcal{Z} : \tau \geq 0\}$  be a family of 1-Lipschitz mappings which satisfy

$$d(x, y) \geq \tau \implies \|\varphi_\tau(x) - \varphi_\tau(y)\| \geq \tau/D$$

for all  $x, y \in X$ .

Consider a finite state space  $\Omega$ , a mapping  $g : \Omega \rightarrow X$ , and a stationary, reversible Markov chain  $\{Z_s\}_{s=0}^\infty$  on  $\Omega$ . We assume that  $Z_0$  is distributed according to the stationary measure. Fix an even number  $t = 2u$ . For each  $j \in \mathbb{Z}$ , let  $\{A_s^{(j)}\}$  and  $\{B_s^{(j)}\}$  be the martingales from Section 3.1 corresponding to the choice  $f = \varphi_{2^j} \circ g : \Omega \rightarrow \mathcal{Z}$ .

From (11), we have  $\varphi_{2^j}(g(Z_0)) - \varphi_{2^j}(g(Z_t)) = A_u^{(j)} - B_u^{(j)}$ . Thus we can write

$$\begin{aligned} \mathbb{E} d(g(Z_0), g(Z_t))^p &= p \int_0^\infty \lambda^{p-1} \cdot \mathbb{P}(d(g(Z_0), g(Z_t)) \geq \lambda) d\lambda \\ &\leq p \sum_{j \in \mathbb{Z}} 2^{(j+1)(p-1)} \cdot \mathbb{P}(d(g(Z_0), g(Z_t)) \geq 2^j) \\ &\leq p \sum_{j \in \mathbb{Z}} 2^{(j+1)(p-1)} \cdot \mathbb{P}\left(\|\varphi_{2^j}(g(Z_0)) - \varphi_{2^j}(g(Z_t))\| \geq \frac{2^j}{D}\right) \\ &= p \sum_{j \in \mathbb{Z}} 2^{(j+1)(p-1)} \cdot \mathbb{P}\left(\|A_u^{(j)} - B_u^{(j)}\| \geq \frac{2^j}{D}\right) \\ &\leq p \sum_{j \in \mathbb{Z}} 2^{(j+1)(p-1)} \cdot \left[\mathbb{P}\left(\|A_u^{(j)}\| > \frac{2^{j-1}}{D}\right) + \mathbb{P}\left(\|B_u^{(j)}\| > \frac{2^{j-1}}{D}\right)\right] \\ &\leq (4D)^p \left(\int_0^\infty y^{p-1} \sup_{j \in \mathbb{Z}} \mathbb{P}\left(\|A_u^{(j)}\| > y\right) dy + \int_0^\infty y^{p-1} \sup_{j \in \mathbb{Z}} \mathbb{P}\left(\|B_u^{(j)}\| > y\right) dy\right) \end{aligned}$$

Now define, for  $1 \leq s \leq u$ ,

$$\begin{aligned} \alpha_s &:= \left(2^{p-1} d(g(Z_{2s}), g(Z_{2s-1}))^p + 2^{p-1} \mathbb{E}[d(g(Z_{2s}), g(Z_{2s-1}))^p \mid Z_{2s-1}]\right)^{1/p} \\ \beta_s &:= \left(2^{p-1} d(g(Z_{t-2s+1}), g(Z_{t-2s}))^p + 2^{p-1} \mathbb{E}[d(g(Z_{t-2s+1}), g(Z_{t-2s}))^p \mid Z_{t-2s}]\right)^{1/p}, \end{aligned}$$

Then, using stationarity, we have the bounds

$$\mathbb{E} \left[ \sum_{s=1}^u \alpha_s^p \right], \mathbb{E} \left[ \sum_{s=1}^u \beta_s^p \right] \leq 4u \mathbb{E} d(g(Z_0), g(Z_1))^p = 2t \mathbb{E} d(g(Z_0), g(Z_1))^p. \quad (14)$$

Additionally, by (12) and (13) and the fact that  $\varphi_{2^j}$  is 1-Lipschitz for each  $j \in \mathbb{Z}$ , for the range  $1 \leq s \leq u$ , we have

$$\begin{aligned} \|A_s^{(j)} - A_{s-1}^{(j)}\| &\leq \alpha_s \\ \|B_s^{(j)} - B_{s-1}^{(j)}\| &\leq \beta_s. \end{aligned}$$

We can thus apply Lemma 2.6 to conclude that

$$\begin{aligned} \int_0^\infty y^{p-1} \sup_{j \in \mathbb{Z}} \mathbb{P}\left(\|A_u^{(j)}\| > y\right) dy &\leq K \mathbb{E} \left[ \sum_{s=1}^u \alpha_s^p \right] \\ \int_0^\infty y^{p-1} \sup_{j \in \mathbb{Z}} \mathbb{P}\left(\|B_u^{(j)}\| > y\right) dy &\leq K \mathbb{E} \left[ \sum_{s=1}^u \beta_s^p \right], \end{aligned}$$

where  $K = O(S_p(\mathcal{Z})^{p/2})$ . Combining these estimates with (14) and our previous discussion, for every even time  $t$ , we have

$$\mathbb{E} \left[ d(g(Z_0), g(Z_t))^p \right] \leq 4K(4D)^p t \mathbb{E} \left[ d(g(Z_0), g(Z_1))^p \right].$$

Finally, if  $t$  is odd, then

$$\begin{aligned} \mathbb{E} \left[ d(g(Z_0), g(Z_t))^p \right] &\leq 2^{p-1} \mathbb{E} \left[ d(g(Z_0), g(Z_{t-1}))^p \right] + 2^{p-1} \mathbb{E} \left[ d(g(Z_{t-1}), g(Z_t))^p \right] \\ &\leq 8K(4D)^p t \mathbb{E} \left[ d(g(Z_0), g(Z_1))^p \right]. \end{aligned}$$

Thus  $(X, d)$  has Markov type  $p$  with constant  $M_p(X) = O(DS_p(\mathcal{Z}))$ .  $\square$

### 3.3 Random partitions

We now recall that planar metrics, doubling metrics, and more general families of metric spaces admit threshold embeddings into Hilbert space, and therefore, by our main theorem, have Markov type 2.

Let  $(X, d)$  be a metric space. If  $P$  is a partition of  $X$  and  $x \in X$ , we will write  $P(x)$  for the unique set of  $P$  containing  $x$ . A *random partition* of  $X$  is a probability space  $(\Omega, \Sigma, \mu)$ , together with a mapping  $\omega \mapsto P_\omega$  which associates to every  $\omega \in \Omega$  a partition  $P_\omega$  of  $X$ . We will use  $\mathcal{P}$  to denote such a random partition.

To sidestep issues of measurability, we will assume that every random partition  $\mathcal{P}$  is supported on only countably many partitions, each of which is composed of only countably many sets of  $X$ . This presents no difficulty because it will hold in all the cases that arise. We refer to [LN06] for a more detailed discussion.

We say that  $\mathcal{P}$  is  $\Delta$ -bounded if

$$\mathbb{P}[\forall S \in \mathcal{P}, \text{diam}(S) \leq \Delta] = \mu(\{\omega : \forall S \in P_\omega, \text{diam}(S) \leq \Delta\}) = 1.$$

We say that  $\mathcal{P}$  is  $(\varepsilon, \delta, \Delta)$ -padded if  $\mathcal{P}$  is  $\Delta$ -bounded, and for all  $x \in X$ ,

$$\mathbb{P}[B(x, \varepsilon\Delta) \subseteq \mathcal{P}(x)] = \mu(\{\omega : B(x, \varepsilon\Delta) \subseteq P_\omega(x)\}) \geq \delta,$$

where  $B(x, R) = \{y \in X : d(x, y) \leq R\}$  denotes the closed ball around  $x$ . The next theorem follows from the techniques of [Rao99] and [LMN05]. The result is well-known, but we could not find the theorem stated explicitly, so we prove it here.

**Theorem 3.2.** *Consider a metric space  $(X, d)$  and numbers  $\varepsilon, \delta > 0$ . Suppose that for every  $\Delta > 0$ ,  $X$  admits an  $(\varepsilon, \delta, \Delta)$ -padded random partition. Then  $(X, d)$   $K$ -threshold embeds into Hilbert space, where  $K \leq \frac{4}{\varepsilon\sqrt{\delta}}$ .*

*Proof.* Fix  $\tau > 0$  and put  $\Delta = \tau/2$ . We may assume that  $\Delta < \text{diam}(X)$ . Let  $\mathcal{P}$  be a  $(\varepsilon, \delta, \Delta)$ -padded random partition with associated probability space  $(\Omega, \Sigma, \mu)$ . For each set  $S$  which occurs as a member of  $\bigcup_{\omega \in \Omega} P_\omega$ , let  $\sigma_S$  be an independent Bernoulli  $\{0, 1\}$  random variable. We use  $(\Omega', \mathbb{P})$  to denote the product space encompassing  $(\Omega, \mu)$  and  $\{\sigma_S\}$ . Finally, consider the map  $\varphi_\tau : X \rightarrow L_2(\Omega', \mathbb{P})$  given by

$$\varphi_\tau(x) = \sigma_{\mathcal{P}(x)} \cdot d(x, X \setminus \mathcal{P}(x)).$$

Observe that for  $x, y \in X$ , we have

$$\|\varphi_\tau(x) - \varphi_\tau(y)\|_{L_2(\Omega', \mathbb{P})}^2 = \mathbb{E} |\sigma_{\mathcal{P}(x)} \cdot d(x, X \setminus \mathcal{P}(x)) - \sigma_{\mathcal{P}(y)} \cdot d(y, X \setminus \mathcal{P}(y))|^2.$$

First, we argue that  $\varphi_\tau$  is 1-Lipschitz. This follows because if  $\mathcal{P}(x) = \mathcal{P}(y)$ , then

$$|d(x, X \setminus \mathcal{P}(x)) - d(y, X \setminus \mathcal{P}(y))| = |d(x, X \setminus \mathcal{P}(x)) - d(y, X \setminus \mathcal{P}(x))| \leq d(x, y).$$

On the other hand, if  $\mathcal{P}(x) \neq \mathcal{P}(y)$ , then

$$d(x, X \setminus \mathcal{P}(x)), d(y, X \setminus \mathcal{P}(y)) \leq d(x, y).$$

Finally, assume that  $d(x, y) \geq \tau$  which implies that  $d(x, y) > \Delta$ . Since  $\mathcal{P}$  is  $\Delta$ -bounded, we have  $\mathbb{P}[\mathcal{P}(x) \neq \mathcal{P}(y)] = 1$ . Therefore,

$$\begin{aligned} & \mathbb{E} |\sigma_{\mathcal{P}(x)} \cdot d(x, X \setminus \mathcal{P}(x)) - \sigma_{\mathcal{P}(y)} \cdot d(y, X \setminus \mathcal{P}(y))|^2 \\ & \geq \mathbb{P}[B(x, \varepsilon \Delta) \subseteq \mathcal{P}(x), \sigma_{\mathcal{P}(x)} = 1, \sigma_{\mathcal{P}(y)} = 0] \cdot \varepsilon^2 \Delta^2 \\ & \geq \frac{\delta}{4} \varepsilon^2 \Delta^2. \end{aligned}$$

It follows that  $d(x, y) \geq \tau \implies \|\varphi_\tau(x) - \varphi_\tau(y)\|_{L_2(\Omega', \mathbb{P})} \geq \frac{\sqrt{\delta \varepsilon} \tau}{4}$ , completing the proof.  $\square$

The next series of results follow from Theorem 3.2 combined with Theorem 3.1 and the existence of padded random partitions. The references for these partitions are [KPR93] for (i) and (iii), [LS10] for (ii), [GKL03] for (iv), and [NS11] for (v). We refer to [LN05] for a treatment of (i), (iii), and (iv).

**Corollary 3.3.** *There is an absolute constant  $K > 0$  such that the following results hold true.*

- i) *If  $(X, d)$  is a planar graph metric, then  $M_2(X) \leq K$ .*
- ii) *If  $(X, d)$  is a metric on a graph of (orientable) genus  $g > 1$ , then  $M_2(X) \leq K \log g$ .*
- iii) *If  $(X, d)$  is a metric on a graph which excludes  $K_h$  as a minor, then  $M_2(X) \leq Kh^2$ .*
- iv) *If  $(X, d)$  is  $\lambda$ -doubling for some  $\lambda \geq 1$ , then  $M_2(X) \leq K \log(1 + \lambda)$ .*
- v) *If  $(X, d)$  has finite Assouad-Nagata dimension, then  $X$  has Markov type 2.*

## 4 Dimension reduction for martingales in smooth Banach spaces

We now present a somewhat weaker form of martingale dimension reduction for  $p$ -uniformly smooth Banach spaces. Consider  $p \in (1, 2]$  and let  $\mathcal{Z}$  be a  $p$ -uniformly smooth Banach space with smoothness constant  $S_p(\mathcal{Z})$ . It is known that for every  $z \in \mathcal{Z}$ , there exists a unique functional  $J_z \in \mathcal{Z}^*$  such that the following conditions hold.

- i)  $\|J_z\| = \|z\|^{p-1}$
- ii)  $\langle J_z, z \rangle = \|z\|^p$ .

iii) There is a constant  $C = O(S_p(\mathcal{Z})^p)$  such that for all  $x, y \in \mathcal{Z}$ ,

$$\|x + y\|^p \leq \|x\|^p + p\langle J_x, y \rangle + C\|y\|^p. \quad (15)$$

See, for instance, [XR91, Eq. (3.8)] and [Chi09, Cor 4.17]. We now prove Lemma 1.16; we reproduce the statement here for the sake of the reader.

**Lemma 4.1.** *For  $p \in (1, 2]$ , the following holds. Let  $\mathcal{Z}$  be a  $p$ -uniformly smooth Banach space and let  $\{M_t\}$  be a  $\mathcal{Z}$ -valued martingale with respect to the filtration  $\{\mathcal{F}_t\}$ . Then there exists an  $\mathbb{R}^2$ -valued martingale  $\{N_t\}$  and a constant  $K > 0$  such that for any time  $t \geq 0$ , the following holds.*

$$i) \quad \|M_t - M_0\|^p \preceq \|N_t - N_0\|_2^2, \text{ and}$$

$$ii) \quad \|N_{t+1} - N_t\|_2^2 \preceq K (\|M_{t+1} - M_t\|^p + \mathbb{E}[\|M_{t+1} - M_t\|^p \mid \mathcal{F}_{t-1}]),$$

where  $K = O(S_p(\mathcal{Z})^p)$ .

*Proof.* Suppose  $\{M_t\}$  is a martingale with respect to the filtration  $\{\mathcal{F}_t\}$ . We may assume that  $\|M_0\| = 0$ . Let  $\{\varepsilon_t\}$  be an independent, i.i.d. sequence of random signs. For  $z \in \mathbb{R}^2$ , let  $z^\perp \in \mathbb{R}^2$  denote a unit vector perpendicular to  $z$ . We define an  $\mathbb{R}^2$ -valued process  $\{N_t\}$  with  $N_0 = (0, 0)$ . Let  $C$  be the constant from (15). For  $t \geq 1$ , we put

$$\begin{aligned} N_t = N_{t-1} & \left( 1 + \frac{p \langle J_{M_{t-1}}, M_t - M_{t-1} \rangle \mathbf{1}_{A_{t-1}} - \delta_{t-1}}{\|N_{t-1}\|_2^2} \right) \\ & + \varepsilon_t N_{t-1}^\perp \left( \sqrt{C + p} \|M_t - M_{t-1}\|^{p/2} + \sqrt{p} (\mathbb{E}[\|M_t - M_{t-1}\|^p \mid \mathcal{F}_{t-1}])^{1/2} \right). \end{aligned}$$

where  $A_{t-1}$  is the event  $\{\|M_t - M_{t-1}\|^p \leq \|N_{t-1}\|_2^2\}$  and  $\delta_{t-1} = \mathbb{E}[\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle \mathbf{1}_{A_{t-1}} \mid \mathcal{F}_{t-1}]$ . From the definition of  $\delta_{t-1}$  and the presence of  $\varepsilon_t$ , it is clear that  $\{N_t\}$  is a martingale.

We will prove claim (i) by induction. It holds trivially for  $t = 0$ . Now assume that  $\|M_{t-1}\|^p \leq \|N_{t-1}\|_2^2$  for some  $t > 1$ . Using the definition of  $A_{t-1}$ , property (i) of the  $J_z$  functional, and our inductive assumption, we have

$$\begin{aligned} |\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle \mathbf{1}_{A_{t-1}^c}| & \leq \|M_{t-1}\|^{p-1} \cdot \|M_t - M_{t-1}\| \mathbf{1}_{A_{t-1}^c} \\ & \leq \|N_{t-1}\|_2^{2(p-1)/p} \cdot \|M_t - M_{t-1}\| \mathbf{1}_{A_{t-1}^c} \\ & \leq \|M_t - M_{t-1}\|^p. \end{aligned} \quad (16)$$

Since the martingale property implies  $\mathbb{E}[\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle \mid \mathcal{F}_{t-1}] = 0$ , we have

$$|\delta_{t-1}| = \left| \mathbb{E}[\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle \mathbf{1}_{A_{t-1}^c} \mid \mathcal{F}_{t-1}] \right| \leq \mathbb{E}[\|M_t - M_{t-1}\|^p \mid \mathcal{F}_{t-1}]. \quad (17)$$

Now we apply property (iii) of the  $J_z$  functional to obtain

$$\|M_t\|^p \leq \|M_{t-1}\|^p + p\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle + C\|M_t - M_{t-1}\|^p. \quad (18)$$

On the other hand, using the definition of  $N_t$ , we have

$$\begin{aligned} \|N_t\|_2^2 & \geq \|N_{t-1}\|_2^2 + p(\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle \mathbf{1}_{A_{t-1}} - \delta_{t-1}) \\ & \quad + (C + p)\|M_t - M_{t-1}\|^p + p\mathbb{E}[\|M_t - M_{t-1}\|^p \mid \mathcal{F}_{t-1}] \\ & \stackrel{(17)}{\geq} \|N_{t-1}\|_2^2 + p\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle \mathbf{1}_{A_{t-1}} + (C + p)\|M_t - M_{t-1}\|^p \\ & \stackrel{(16)}{\geq} \|N_{t-1}\|_2^2 + p\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle + C\|M_t - M_{t-1}\|^p \\ & \geq \|M_{t-1}\|^p + p\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle + C\|M_t - M_{t-1}\|^p, \end{aligned}$$

where in the final line we have used the inductive hypothesis. Combined with (18), this yields  $\|M_t\|^p \leq \|N_t\|_2^2$ , verifying claim (i).

We proceed to verify claim (ii). From the definition of  $A_{t-1}$  and the inductive hypothesis, we have

$$\begin{aligned} |\delta_{t-1}| &= |\mathbb{E} [\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle \mathbf{1}_{A_{t-1}} \mid \mathcal{F}_{t-1}]| \\ &\leq \mathbb{E} [\|M_{t-1}\|^{p-1} \|M_t - M_{t-1}\| \mathbf{1}_{A_{t-1}} \mid \mathcal{F}_{t-1}] \\ &\leq \|N_{t-1}\|_2^2. \end{aligned} \tag{19}$$

Additionally, using property (i) of the  $J_z$  functional, the inductive hypothesis, and the definition of  $A_{t-1}$ , we have

$$\begin{aligned} \frac{\langle J_{M_{t-1}}, M_t - M_{t-1} \rangle^2 \mathbf{1}_{A_{t-1}}}{\|N_{t-1}\|_2^2} &\leq \frac{\|M_{t-1}\|^{2(p-1)} \|M_t - M_{t-1}\|^2 \mathbf{1}_{A_{t-1}}}{\|N_{t-1}\|_2^2} \\ &\leq \frac{\|M_t - M_{t-1}\|^2 \mathbf{1}_{A_{t-1}}}{\|N_{t-1}\|_2^{4(2-p)/p}} \\ &\leq \|M_t - M_{t-1}\|^p. \end{aligned}$$

Finally, using (17) and (19) yields

$$\frac{|\delta_{t-1}|^2}{\|N_t\|_2^2} \leq \mathbb{E} [\|M_t - M_{t-1}\|^p \mid \mathcal{F}_{t-1}].$$

Combining the preceding two inequalities with the definition of  $N_t$ , we arrive at

$$\|N_t - N_{t-1}\|_2^2 \leq O(C) (\|M_t - M_{t-1}\|^p + \mathbb{E} [\|M_t - M_{t-1}\|^p \mid \mathcal{F}_{t-1}]),$$

completing the verification of claim (ii). □

## 5 Non-linear analogs of Kwapien's theorem

First, we recall a few definitions. A *uniform embedding* between metric spaces is an invertible mapping such that both it and its inverse are uniformly continuous. A mapping  $f : X \rightarrow L_2$  is a *coarse embedding* if there are non-decreasing maps  $\alpha, \beta : [0, \infty) \rightarrow [0, \infty)$  such that  $\beta(t) \rightarrow \infty$  and for all  $x, y \in X$ ,

$$\beta(d(x, y)) \leq \|f(x) - f(y)\|_2 \leq \alpha(d(x, y)).$$

We begin by proving Theorem 1.9 which we restate for convenience.

**Theorem 5.1.** *A Banach space  $Z$  threshold-embeds into Hilbert space if and only if  $Z$  is linearly isomorphic to a Hilbert space.*

As a tool, we will need the next result which follows from techniques of [Ass83] and whose proof we defer for a moment. We recall that if  $(X, d)$  is a metric space and  $\varepsilon \in (0, 1]$ , we use  $(X, d^\varepsilon)$  to denote the metric space with distance  $d^\varepsilon(x, y) = d(x, y)^\varepsilon$ .

**Lemma 5.2** ([Ass83]). *If  $(X, d)$  threshold-embeds into  $L_2$  then for every  $\varepsilon \in (0, 1)$ , the space  $(X, d^{1-\varepsilon})$  bi-Lipschitz embeds into a Hilbert space.*



*Proof of Theorem 5.1.* Since  $Z$  threshold-embeds into Hilbert space, it has Markov type 2 by Theorem 3.1, hence it also has linear type 2 [Bal92]. One also has the following alternate and simpler line of argument:  $Z$  has Enflo type 2 by Proposition 1.10, hence it also has linear type 2 [Enf70].

On the other hand, by Lemma 5.2,  $Z$  uniformly embeds into Hilbert space, and thus by [AMM85] (see also [BL00, Cor. 8.17]),  $Z$  has cotype 2. Now Kwapien's theorem [Kwa72] implies that  $Z$  is isomorphic to a Hilbert space.  $\square$

We now prove Lemma 5.2. The argument is folklore, but we could not find it written explicitly.

*Proof of Lemma 5.2.* Suppose that  $\{\varphi_\tau : X \rightarrow L_2 : \tau > 0\}$  is a  $K$ -threshold-embedding of  $X$  into  $L_2$ . Using [MN04, Lem. 5.2], we may assume that  $\{\varphi_\tau : X \rightarrow L_2 : \tau > 0\}$  is a  $2K$ -threshold-embedding with the additional property that  $\sup_{x \in X} \|\varphi_\tau(x)\| \leq \tau$  for all  $\tau > 0$ . By scaling, we may assume that each  $\varphi_\tau$  is 1-Lipschitz.

Let  $\{e_n\}_{n \in \mathbb{Z}}$  be an orthonormal basis for  $\ell_2$  and define the map  $\Phi : X \rightarrow L_2 \otimes \ell_2$  by

$$\Phi(x) = \sum_{n \in \mathbb{Z}} 2^{-\varepsilon n} \varphi_{2^n}(x) \otimes e_n.$$

Fix  $x, y \in X$  and let  $m \in \mathbb{Z}$  be such that  $d(x, y) \in [2^m, 2^{m+1})$ . On the one hand, we have

$$\|\Phi(x) - \Phi(y)\|_{L_2 \otimes \ell_2} \geq 2^{-\varepsilon m} \|\varphi_{2^m}(x) - \varphi_{2^m}(y)\| \geq 2^{-\varepsilon m} \frac{2^m}{2K} \geq \frac{d(x, y)^{1-\varepsilon}}{4K}.$$

On the other hand,

$$\begin{aligned} \|\Phi(x) - \Phi(y)\|_{L_2 \otimes \ell_2}^2 &= \sum_{n \in \mathbb{Z}} 2^{-2\varepsilon n} \|\varphi_{2^n}(x) - \varphi_{2^n}(y)\|^2 \\ &\leq \sum_{n < m} 2^{-2\varepsilon n} 2^{2n} + 2^{2(m+1)} \sum_{n \geq m} 2^{-2\varepsilon n} \\ &\leq O\left(\frac{1}{\varepsilon}\right) 2^{2(1-\varepsilon)m} \\ &\leq O\left(\frac{1}{\varepsilon}\right) d(x, y)^{2(1-\varepsilon)}. \end{aligned}$$

$\square$

We can now move onto some partial non-linear analogs of Kwapien's theorem. By [AMM85] and [Ran06], respectively, it is known that if a Banach space  $Z$  admits a uniform or coarse embedding into Hilbert space, then  $Z$  has cotype 2. On the other hand, there do exist Banach spaces of cotype 2 that do not uniformly or coarse embed into Hilbert space (e.g.  $C_1$  the Schatten trace class); see [BL00, §8.2].

**Theorem 5.3.** *Suppose that a Banach space  $Z$  admits a coarse or uniform embedding into a Hilbert space. Then a subset  $X \subseteq Z$  has Markov type 2 if and only if  $X$  threshold-embeds into a Hilbert space.*

To prove this, we need a few definitions. Consider a metric space  $(X, d)$ . We recall that an  $\varepsilon$ -net is a maximal subset  $N \subseteq X$  such that  $d(x, y) \geq \varepsilon$  for all  $x, y \in N$ . Say that a map  $f : X \rightarrow L_2$  is  $\tau$ -thresholding if for all  $x, y \in X$ , we have  $d(x, y) \geq \tau \implies \|f(x) - f(y)\| \geq \tau$ . Define the parameter

$$\Lambda_\tau(X, \varepsilon) := \inf \left\{ \sup_{\substack{x, y \in X \\ d(x, y) \geq \varepsilon\tau}} \frac{\varepsilon \|f(x) - f(y)\|_2}{d(x, y)} : f : X \rightarrow L_2 \text{ is } \tau\text{-thresholding} \right\}.$$

Finally, we define  $\Lambda(X, \varepsilon) = \sup_{\tau > 0} \Lambda_\tau(X, \varepsilon)$ . Any separable metric space  $X$  trivially satisfies  $\Lambda(X, \varepsilon) \leq 1$ . We say that  $(X, d)$  is  $\Lambda$ -nontrivial if  $\liminf_{\varepsilon \rightarrow 0} \Lambda(X, \varepsilon) = 0$ . Otherwise, we say that  $(X, d)$  is  $\Lambda$ -trivial. The next result is straightforward.

**Lemma 5.4.** *If  $(X, d)$  is  $\Lambda$ -nontrivial, then so is a subset  $Y \subseteq X$ .*

Now we are in position to use Markov type 2 in conjunction with Ball's extension theorem.

**Lemma 5.5.** *If  $(X, d)$  is  $\Lambda$ -nontrivial, then a subset  $X$  has Markov type 2 if and only if  $X$  threshold embeds into  $L_2$ .*

*Proof.* In light of Theorem 3.1, we need only prove the only if direction. Suppose that  $M_2(X) \leq C$ . By Ball's extension theorem (Theorem 1.2), there exists a constant  $\hat{C} = O(C)$  such that every Lipschitz map from a subset of  $X$  into  $L_2$  admits a Lipschitz extension whose Lipschitz constant is larger by at most a factor of  $\hat{C}$ .

Fix some  $\tau > 0$ . Since  $X$  is  $\Lambda$ -nontrivial, there exists an  $\varepsilon < \frac{1}{4}$  and a mapping  $f : X \rightarrow L_2$  such that for any  $(\varepsilon\tau)$ -net  $N \subseteq X$ , we have  $\|f|_N\|_{\text{Lip}} \leq \frac{1}{8\hat{C}\varepsilon}$  and which satisfies, for all  $x, y \in X$ ,

$$d(x, y) \geq \frac{\tau}{2} \implies \|f(x) - f(y)\| \geq \frac{\tau}{2}. \quad (20)$$

Let  $\tilde{f} : X \rightarrow L_2$  be the extension of  $f|_N$  guaranteed by Ball's extension theorem, so that  $\|\tilde{f}\|_{\text{Lip}} \leq \frac{1}{8\varepsilon}$ .

Fix any  $x, y \in X$  with  $d(x, y) \geq \tau$  and let  $x', y' \in N$  be such that  $d(x, x'), d(y, y') \leq \varepsilon\tau$ . In particular, since  $\varepsilon < \frac{1}{4}$ , we have  $d(x', y') \geq \tau/2$ . Using the triangle inequality yields

$$\begin{aligned} \|\tilde{f}(x) - \tilde{f}(y)\| &\geq \|\tilde{f}(x') - \tilde{f}(y')\| - 2\varepsilon\tau\|\tilde{f}\|_{\text{Lip}} \\ &\geq \|f(x') - f(y')\| - \frac{\tau}{4} \\ &\geq \frac{\tau}{2} - \frac{\tau}{4} = \frac{\tau}{4}, \end{aligned}$$

where in the final line we have used (20). Since  $\tau > 0$  was arbitrary, this proves that  $X$  threshold embeds into  $L_2$ , completing the proof.  $\square$

The next two lemmas, combined with Lemma 5.5, comprise a proof of Theorem 5.3.

**Lemma 5.6.** *If a Banach space  $Z$  admits a coarse embedding into  $L_2$  then  $Z$  is  $\Lambda$ -nontrivial.*

*Proof.* Let  $f : Z \rightarrow L_2$  be a coarse embedding with moduli  $\alpha, \beta$ . Since  $Z$  is a normed space, its metric is convex, and we can assume that  $f$  is Lipschitz for large distances (see, e.g., [BL00, Prop. 1.11]). Thus after rescaling  $f$  (and the moduli  $\alpha, \beta$ ), we may assume that

$$\|x - y\|_Z \geq 1 \implies \|f(x) - f(y)\|_2 \leq \|x - y\|_Z. \quad (21)$$

Now, by homogeneity, to show that  $Z$  is  $\Lambda$ -nontrivial, it suffices to show that for every  $\delta > 0$ , there exist  $\tau, \varepsilon$  such that

$$\Lambda_\tau(Z, \varepsilon) < \delta.$$

To this end, let  $\tau > 0$  be chosen large enough so that  $\beta(\tau) > \delta^{-1}$ , and define  $g = f \cdot \frac{\tau}{\beta(\tau)}$ . Also, we put  $\varepsilon = 1/\tau$ . First, if  $\|x - y\|_Z \geq \tau$ , then  $\|g(x) - g(y)\|_2 \geq \frac{\tau}{\beta(\tau)} \|f(x) - f(y)\|_2 \geq \tau$ , hence  $g$  is  $\tau$ -thresholding.

Next, consider any pair  $x, y \in Z$  with  $\|x - y\|_Z \geq \varepsilon\tau = 1$ . Then,

$$\varepsilon \cdot \frac{\|g(x) - g(y)\|_2}{\|x - y\|_Z} = \frac{\|g(x) - g(y)\|_2}{\tau \|x - y\|_Z} = \frac{\|f(x) - f(y)\|_2}{\beta(\tau) \|x - y\|_Z} \leq \frac{1}{\beta(\tau)} < \delta,$$

where in the penultimate inequality, we have used (21).  $\square$

**Lemma 5.7.** *If a Banach space  $Z$  admits a uniform embedding into  $L_2$  then  $Z$  is  $\Lambda$ -nontrivial.*

*Proof.* Let  $f : Z \rightarrow L_2$  be a uniform embedding. Since  $f^{-1}$  is uniformly continuous, there exists a  $\tau > 0$  such that  $\|x - y\|_Z \geq 1 \implies \|f(x) - f(y)\|_2 \geq \tau$ . By rescaling, we may assume that  $f$  is  $\tau$ -thresholding.

Then by a simple metric convexity argument, we have

$$\sup_{\substack{x, y \in Z \\ \|x - y\|_Z \geq \varepsilon\tau}} \frac{\varepsilon \|f(x) - f(y)\|_2}{\|x - y\|_Z} \leq \sup_{\substack{x, y \in Z \\ \varepsilon\tau/2 \leq \|x - y\|_Z \leq \varepsilon\tau}} \frac{2\|f(x) - f(y)\|_2}{\tau}.$$

The latter quantity goes to 0 as  $\varepsilon \rightarrow 0$ , since  $f$  is uniformly continuous. Hence  $\Lambda_\tau(Z, \varepsilon) \rightarrow 0$ . Thus by homogeneity,  $Z$  is  $\Lambda$ -nontrivial.  $\square$

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